Non-self-averaging Economic Growth: A Criticism of Endogenous Growth Theory

Masanao Aoki*
Department of Economics
University of California, Los Angeles
Fax Number 1-310-825-9528, aoki@econ.ucla.edu;
and
Hiroshi Yoshikawa
Facultly of Economics, Universtiy of Tokyo
Fax Number 81-3-5841-5521, yoshikawa@e.u-tokyo.ac.jp

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Abstract

Using a simple stochastic growth model, this paper demonstrates that the coefficient of variation of aggregate output or GDP does not necessarily go to zero even if the number of sectors or economic agents goes to infinity. This phenomenon known as non-self-averaging in physics occurs in the two-parameter Poisson-Dirichlet model. The implication is that even if the number of economic agents is large, dispersion can remain significant, and, therefore, that we can not legitimately focus on the means of aggregate variables. It, in turn, means that the standard microeconomic foundations based on the representative agent has little value, for they are expected to provide us with dynamics of the means of aggregate variables.

The paper also shows that the two-parameter model exhibits power-law behavior, while the one-parameter version does not.

Key Words: Non-self averaging phenomena, Power laws, Micro-foundations.

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Introduction

The contribution of the literature on endogenous growth ranging from Romer (1986) and Lucas (1988) to Grossman and Helpman (1991), and Aghion and Howitt (1992), has been to *endogenize* the underlying source of sustained growth in per-capita income. The main analytical exercises in these papers are to explicitly consider the optimization by the representative agent in such activities as education, on-the-job training, basic scientific research, and process and product innovations. This approach is not confined to the study of economic growth, but actually originated in the theory of business cycles represented by the real business cycle theory (Kydland and Prescott (1982)):

> "Real business cycle models view aggregate economic variables as the outcomes of the decisions made by many individual agents acting to maximize their utility subject to production possibilities and resource constraints. As such, the models have an explicit and firm foundation in microeconomics. Plosser (1989, p.53)."

This is the basic tenor which applies not only to the theory of business cycles, but also to the endogenous growth literature, or for that matter to the whole macroeconomic theory. Lucas (1987) declared against the old macroeconomics.

> "The most interesting recent developments in macroeconomic theory seem to me describable as the reincorporation of aggregative problems such as inflation and the business cycle within the general framewrok of "microeconomic" theory. If these developments succeed, the term 'macroeconomic' will simply disappear from use and the modifier 'micro' will become superfluous. We will simply speak, as did Smith, Ricardo, Marshall and Walras, of *economic* theory (Lucas, (1987; p.107-108))."

Using a simple growth model, we show that these research programs are misguided. Contrary to Lucas' assertion, we need a defferent approach to macroeconomics from microeconomic theory (Aoki and Yoshikawa (2007)).

Whether in growth or business cycle models, the fundamental cause for often complex optimization exercises is that they are expected to lead us to our better understanding dynamics of the *means* of *aggregate* variables. The standard model thus begins with the analysis of the optimization of the representative agent, and translates it into the analysis of the economy as a whole.

Economists doing these exercises are, of course, well aware that economic agents differ, and that they are subject to idiosyncratic (or microeconomic) shocks. As we will observe in the final section, idiosyncratic shocks are indeed the key factor in Lucas (1972, 73)'s theory of business cycles. However, their analyses presuppose that those microeconomic shocks and differences cancel out each other, and that the behaviors of aggregate variables are represented by their means which, in turn, can be well captured by the analysis based on the representative agent.

The point is best illustrated by the Poisson model which is so widely used in economics ranging from labor search theory to endogenous growth models. Suppose that a constant Poisson parameter, λ , denotes the instantaneous probability that an "event" such as technical progress and job arrival occurs. This probability pertains to one economic agent. Given the same Poisson process with parameter λ for each individual agent, we obtain the Poisson process with the parameter λN for the economy as a whole where there are N economic agents. The mean and the standard deviation of the number of "events" in the macroeconomy are λN and $\sqrt{\lambda N}$, respectively. The coefficient of variation defined as the standard deviation divided by the mean, is, therefore, $\sqrt{\lambda N}/\lambda N = 1/\sqrt{\lambda N}$. Thus, in the Poisson model, when the number of economic agents N becomes large ($N \to \infty$), the coefficient of variation approaches zero.

This property known as *self-averaging* provides us with justification for our concentrating on the means of variables in *macro* models; The macroeconomy consists of a large number of economic agents. Now, because the mean depends basically on λ, it is natural to explore how λ is determined in models. Indeed, in standard models, λ is endogenously determined as an outcome of economic agents' optimization and market equilibrium. The Poisson model is just an example. We all know that a considerable part of every main stream macroeconomics paper is now devoted to this kind of micro optimization exercise.

So far, so good. There is, however, an important point that the standard Poisson model tacitly presumes the representative agent; Economic agents are homogenous in that they face the *same unchanged* instantaneous probability that an "event" occurs to them. Microsoft and small grocery stores on the street face "idyosyncratic" or micro shocks which come from the *same* probability distribution! This crucial assumption is also made in the well-known rational expectations model of Lucas, or for that matter in all micro-founded macroeconomic models. When we drop this crucial assumption, we realize that the standard microedonomic foundations for macroeconomics are actually wholly misguided.

Specifically, using a simple stochastic model, this paper demonstrates that a tacit and yet the fundamental assumption underlying endogenous growth and real business cycle theories, namely the law of large numbers, is not generally tenable. We show that even if the number of economic agents is large, the behavior of the macroeconomy can not be generally well approximated by the means. The implication is that analyses based on the representative agent which generate the means of stochastic time paths of aggregate variables, have little value. Put it another way, the standard micro-foundations are not actually true micro-foundations.

Before we proceed to the model, we explain the notion called "non-self-averaging," the crucial concept for our purpose. The term "non-self-averaging" is extensively used in the physics literature; see Sornette (2000, p.369), but it is not known in economics. "Non-self-averaging" means that a size-dependent (*i.e.* "extensive" in physics) random variable X of the model has the coefficient of variation that does not converge to zero as model size goes to infinity.[1] The coefficient of variation *c.v.* of an extensive random variable, X defined by

$$C.V.(X) = \frac{\sqrt{variance(X)}}{mean(X)}.$$

is normally expected to converge to zero as model size (e.g. the number of economic agents) goes to infinity. In this case, the model is said to be "self-averaging." We have already shown that the popular Poisson model has this self-averaging property. However, this property is not generic at all; In many models, we are actually led to non-self-averaging.

The notion of non-self-averaging is important because non-self-averaging models are sample dependent, and some degree of impreciseness or dispersion remains about the time trajectories even when the number of economic agents go to infinity. This implies that focus on the mean path behavior of macroeconomic variables does not have any scientific justification. It, in turn, means that sophisticated optimization exercises which provide us with information on the means have little value.

In what follows, we first demonstrate this point using the two-parameter Poisson-Dirichlet model. We next show that based on urn models, non-self-averaging is not confined to a particular model which we present in the next section, but is actually quite generic. The final section offers concluding discussion on the implications of non-self-averaging for macroeconomics.

1 Non-self-averaging in a Growth Model

In this section, we present a simple innovation-driven growth model in which aggregate output or GDP is non-self-averaging.

The Model

Following the spirit of endogenous growth, we assume that the economy grows by innovations. Innovations are shochastic events. There are two kinds of innovations in our model. Namely, an innovation, when it occurs, either

[1] This limit is called thermodynamic limit in the physics terminology. The thermodynamic behavior of these two classes of models have been examined in Aoki (2006), and is shown to be qualitatively different betwee these two-classes of models.

raises productivity of one of the existing sectors, or creates a new sector. Thus, the number of sectors is not given, but increases over time.

By the time nth innovation occurs, the total of K_n sectors are formed in the economy wherein the i-th sector has experienced n_i innovations ($i = 1, 2, \ldots, K_n$). By definition, the following equality holds:

$$n_1 + n_2 + \cdots + n_k = n \tag{1}$$

when $K_n = k$. If n-th innovation creates a new sector (sector k), then $n_k = 1$.

The aggregate output or GDP is denoted by Y_n when n innovations have occured. Y_n is simply the sum of outputs in all the sectors, y_i.

$$Y_n = \sum_{i}^{K_n} y_i. \tag{2}$$

Output in secotr i grows owing to innovations which stochastically occur in that sector. Specifically, we assume

$$y_i = \eta\gamma^{n_i}. \qquad (\eta > 0, \gamma > 1) \tag{3}$$

It is convenient at this point to rewrite equation (1) as follows.

$$n = \sum_{j}^{n} j a_j(n) \tag{4}$$

In equation (4), $a_j(n)$ is the number of sectors where j innovations have occurred. The vector consisting of $a_j(n)$, $a(n)$ is called *partition vector*[2]. With this partition vector, K_n can be expressed as the sum of all the components of the partition vector

$$K_n = \sum_{j}^{n} a_j(n). \tag{5}$$

Using the following approximation

$$\gamma^{n_i} = exp(n_i \ln \gamma)) \approx 1 + \ln(\gamma) n_i,$$

we can rewrite equation (3) as

$$y_i = \eta + \eta \ln(\gamma) n_i. \tag{6}$$

Thus, from equations (1), (2), (4), (5) and (6), we obtain

$$Y_n \approx K_n + \beta \sum_{j}^{n} j a_j(n) = \sum_{j}^{n} (1 + \beta j) a_j(n). \tag{7}$$

where $\beta = \ln(\gamma) > 0$. Here, without loss of generality, we assume that η in equation (3) is one. Obviously, the behavior of the aggregate output, Y_n depends on how innovations occur.

[2]See chapter 2 of Aoki and Yoshikawa (2007) for partition vector.

The Poisson-Dirichlet Distribution of Innovations

We now describe how innovations stochastically occur in the model. An innovation follows *the two-parameter Poisson-Dirichlet (PD) distribution.*[3]

Given the two-parameter Poisson-Dirichlet distribution, when there are k clusters of sizes n_i, $(i = 1, 2, \ldots, k)$, and $n = n_1 + n_2 + \cdots + n_k$, assume that an innovation occurs in one of the existing sectors of size n_i with probability rate p_i:

$$p_i = \frac{n_i - \alpha}{n + \theta}. \tag{8}$$

The "size" of sector i, n_i is equal to the number of innovations that have already occurred in sector i. The two parameters α and θ satisfy the following conditions:

$$\theta + \alpha > 0, \quad \text{and} \quad 0 < \alpha < 1.$$

With $\alpha = 0$ there is a single parameter θ, and the distribution boils down to the one-parameter Poisson-Dirichlet distribution, $PD(\theta)$. Since p_i denotes the probability that an innovation occurs in one of the existing sectors. Thus, a new sector emerges with probability rate[4] p:

$$p = 1 - \sum_{1}^{k} \frac{n_i - \alpha}{n + \theta} = \frac{\theta + k\alpha}{n + \theta}. \tag{9}$$

It is important to note that in this model, sectors are *not* homogeneous with respect to the probability that an innovation occurs. The larger sector i is, the greater becomes the probability that an innovation occurs in sector i. Moreover, these probabilities *change endogenously* as n_i changes over time.

In the one-parameter case where $\alpha = 0, p = \theta/(\theta + n)$ is a probability rate that the $(n+1)$th innovation generates a new sector while $\sum_i p_i = n/(\theta + n)$

[3]Kingman invented the one-parameter Poisson-Dirichlet distribution to describe random partitions of populations of heterogeneous agents into distinct clusters. The one-parameter Poisson-Dirichlet model is also known as Ewens model, (Ewens (1972)); See Aoki (2000a, 2000b) for further explanation. The one-parameter model was then extended to the two-parameter Poisson-Dirichlet distributions by Pitman; See Kingman (1993), Carlton (1999), Feng and Hoppe(1998), Pitman (1999, 2002), and Pitman and Yor (1996), among others. Aoki (2006) has shown that the two-parameter Poisson-Dirichlet models are qualitatively different from the one-parameter version because the former is not self-averaging while the latter is. These models are therefore not exponential growth models familiar to economists but they belong to a broader class of models without steady state constant exponential growth rate. None of the previous works, however, have comparatively examined the asymptotic behavior of the coefficient of variation of these two classes of models.

[4]Probabilities of new types entering Ewens model, are discussed in Aoki (2002, Sec.10.8, App. A.5).

is the probability that the innovation raises productivity of one of n existing sectors.

In the two-parameter Poisson-Dirichlet distribution, the probability that the number of sectors increases by one conditional on $K_n = k$ is given by

$$\Pr(K_{n+1} = k+1 | K_1, \ldots, K_n = k) = p = \frac{\theta + k\alpha}{n+\theta}, \tag{10}$$

and the probability that the number of sectors remain unchanged is

$$\Pr(K_{n+1} = k | K_1, \ldots, K_n = k) = \sum_i p_i = \frac{n - k\alpha}{n+\theta}. \tag{11}$$

Because the following inequality holds:

$$\frac{\theta + k\alpha}{n+\theta} > \frac{\theta}{n+\theta},$$

we observe that the probability that a new sector emerges is higher in the two-parameter *PD* model than in the one-parameter *PD* model.

We show that this two-parameter PD model is non-self-averaging. It is interesting to observe that the one parameter PD model (where α=0) is self-averaging. Before we proceed, it may be helpful to say a few words why the two-parameter PD model is non-self-averaging. The answer lies in (10) and (11). In this model, innovations occur in one of the two different types of sectors, one, the new type and the other, known or pre-existing types. The probability that an innovation generates a new sector is $(\theta + K_n\alpha)/(n+\theta)$, and the probability that an inovation occurs in one of the existing sectors is $(n - K_n\alpha)/(n+\theta)$, where K_n is the number of types of sectors in the model by the time n innovations have occurred. These probabilities and their ratio vary **endogenously**, depending on the histories of how innovations occured. In other words, the mix of old and new sectors evolve endogenously, and is path-dependent. This is the reason why non-self-averaging emerges in the two-parameter PD model. We note that in one parameter PD model in which $\alpha = 0$, two probabilities (10) and (11) becomes independent of K_n, and that the model becomes self-averaging.

Now, the standard endogenous growth literature focuses on profit motives for innovations. The name "endogenous growth" comes from explicit analysis of innovations as outcomes of profit-seeking activities. There is no denying that innovations are at least partly outcomes of intentional profit-seeking activities. However, we skip such analysis in the present analysis. The fundamental reason is that aggregate output, Y_n, is non-self averaging. To explain this point in detail is, in fact, the purpose of this paper.

GDP is Non-self Averaging

Given the model, we are interested in the behavior of GDP, namely Y_n. Specifically, we would like to see whether or not Y_n is self-averaging. Toward this goal, we first nomalize Y_n by n^α. Then, from equation (7), we obtain

$$\frac{Y_n}{n^\alpha} = \Sigma_j^n (1+\beta j) \frac{a_j(n)}{n^\alpha} \tag{12}$$

Next, to show that Y_n is non-self-averaging, we first define partial sums of K_n and Y_n up to $l(< n)$, $K_n(1,l)$ and $Y_n(1,l)$ as follows:

$$K_n(1,l) = \sum_{j=1}^{l} a_j(n), \tag{13}$$

and

$$Y_n(1,l) = K_n(1,l) + \beta \hat{Y}_n(1,l), \tag{14}$$

where

$$\hat{Y}_n(1,l) = \sum_{j=1}^{l} j a_j(n). \tag{15}$$

Yamato and Sibuya (2000; p.7, their Prop. 4.1 and 4.2) showed that given l, $K_n(1,l)/n^\alpha$ and $\hat{Y}_n(1,l)/n^\alpha$ converge in distribution as n grows unboundedly as follows

$$\frac{K_n(1,l)}{n^\alpha} \to^d C_1(l). \tag{16}$$

They have established the following theorem. It is basic for our analysis.

Theorem (Yamato and Sibuya (2000)) :

Given the two-parameter Poisson-Dirichlet model, with $0 < \alpha < 1$, and a positive integer j, $a_j(n)$ normalized by n^α converges in distribution to a random variable L. That is,

$$\frac{a_j(n)}{n^\alpha} \to^d f_j L \qquad (j = 1, 2, \ldots). \tag{17}$$

Here, f_j is defined by

$$f_j := \frac{\alpha(1-\alpha)^{[j-1]}}{j!}, \tag{18}$$

where $[j]$ denotes an ascending factorial:

$$x^{[j]} = x(x+1)\ldots(x+j-1). \tag{19}$$

The random variable L in (13) is known to have probability density function $g_{\alpha,\theta}(x)$:

$$g_{\alpha,\theta}(x) = \frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)} x^{\theta/\alpha} g_\alpha(x). \tag{20}$$

$g_\alpha(x)$ in (20) is called Mittag-Leffler density function [5].

This theorem allows us to conclude that given equation (12), Y_n normalized by n^α converges in distribution to L.

$$\frac{Y_n}{n^\alpha} = \Sigma_j^n (1+\beta j) \frac{a_j(n)}{n^\alpha} \longrightarrow^d \Sigma_j^n (1+\beta j) f_j L \tag{21}$$

We note that because $\Sigma_j^n(1+\beta j)f_j$ is a constant, the coefficient of variation of Y_n/n^α, $C.V.(Y_n/n^\alpha)$, is equal to that of L. All we need to know are, therefore, the moments of L.

The function introduced in (20), $g_\alpha(x)$,has the property that its p th moment, $p > -1$, is given by

$$\int_0^\infty x^p g_\alpha(x) dx = \frac{\Gamma(p+1)}{\Gamma(p\alpha+1)}. \tag{22}$$

Thus, we obtain the first and second moments of L, $E_{\alpha,\theta}(L)$ and $E_{\alpha,\theta}(L^2)$ as follows:

$$E_{\alpha,\theta}(L) = \frac{\Gamma(\theta+1)}{\alpha\Gamma(\theta+\alpha)}, \tag{23}$$

and

$$E_{\alpha,\theta}(L^2) = \frac{(\theta+\alpha)\Gamma(\theta+1)}{\alpha^2\Gamma(\theta+2\alpha)}. \tag{24}$$

The variance of L is, therefore,

$$var(L) = E_{\alpha,\theta}(L^2) - [E_{\alpha,\theta}(L)]^2 = \gamma_{\alpha,\theta}\frac{\Gamma(\theta+1)}{\alpha^2}, \tag{25}$$

where

$$\gamma_{\alpha,\theta} := \frac{\theta+\alpha}{\Gamma(\theta+2\alpha)} - \frac{\Gamma(\theta+1)}{[\Gamma(\theta+\alpha)]^2}. \tag{26}$$

It is important to notice that $\gamma_{\alpha,\theta}$ is zero when α is zero, because the non-self averaging property is due to this fact.

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[5] See Blumenfeld and Mandelbrot (1997), Erdely, A., W.Magnus, F Oberhettinger, F.G. Tricomi (1953-1954), or Pitman (1999) on Mittag-Leffler function.

The coefficient of variation of L is given by

$$C.V.(L) = \frac{\sqrt{var(L)}}{E_{\alpha,\theta}(L)} = \sqrt{\frac{\gamma_{\alpha,\theta}}{\Gamma(\theta+1)}}\Gamma(\theta+\alpha). \tag{27}$$

By the theorem of Yamato and Sibuya, we obtain

$$C.V.\left(\frac{Y_n}{n^\alpha}\right) \to C.V.(L). \tag{28}$$

Because of (27) and (28), we finally obtain

$$C.V.\left(\frac{Y_n}{n^\alpha}\right) \to \sqrt{\frac{\gamma_{\alpha,\theta}}{\Gamma(\theta+1)}}\Gamma(\theta+\alpha). \tag{29}$$

With non-zero α, the right-hand side of (25) does not approach zero even when n goes to infinity. Thus, we have established the following proposition.

Proposition
In the two-parameter Poisson-Dirichlet model, the aggregate output Y_n is non-self averaging.

Concluding Discussion

In traditional microeconomic foundations of macroeconomics, it is taken for granted that as the number of agents goes to infinity, any unpleasant fluctuations vanish and well defined deterministic macroeconomic relations prevail.

That is why economists focus on the optimizing behavior of the representative agent. However, if model is non-self-averaging, dispersion remains even if the number of economic agents become infinite. It means that we can not focus on means. This, in turn, means that sophisticated optimization and market equilibrium exercises which provide us with dynamics of means have, in fact, little value.

Take Aghion and Howitt (1992) as a primary example. In their paper, the Poisson arrival rate of innovations is assumed to be an increasing function of the economy-wide flow of skilled labor used in research, n; Namely, they have $\lambda\phi(n)$ and $\phi'(n) > 0$. Profit-maximizing monopolist' decision on research and development on the one hand and perfect foresight market equilibrium on the other determine n. They focus on $\hat{n}$ in a stationary equilibrium. Then, the means and variance of the economy's growth rate are $\lambda\phi(\hat{n})\ln\gamma$ and $\lambda\phi(\hat{n})(\ln\gamma)^2$, respectively. They do not calculate the coefficient of variation of the growth rate. If they did, they would have discovered that the coefficient of variation of the growth rate remains non zero even if n becomes infinite. The only way for it to vanish as the model size or the number of observation grows is to have their Poisson parameter $\lambda\phi(\hat{n})$ go to infinity, a phathological explosive case. Unfortunately it did not catch the attention of the economic audiences including the authors! Because the Aghion-Howitt (1992) model is non-self averaging, their micro foundations exercises — monopolists' decision on R & D and perfect foresight market equilibrium — actually have little value. Solow (2000) suggests that we might reasonably separate macroeconomic growth theory from microeconomic analysis of technical progress.

> "It may be too strong a statement, but only a little too strong, to suggest that growth theory "proper" is the study of the long-run behavior of an economy conditional on $A(t)$. But then there is a separate, though closely related, field of study that is concerned with $A(t)$ itself, or more generally with the understanding of the process of technological change. It goes without saying that the results of this second branch of economics will be of central importance to growth theory. One of the advantages of this distinction is that the economics of technical change will certainly involve considerations — about industrial organization, management, practices, and other such things — that have little in common with the macroeconomics of growth, but are essential to the economics of technology. (Solow (2000), p.101)"

Non-self-averaging leads us to the same conclusion.

We have shown that in general, economic growth is non-self-averaging, and, therefore, that exclusive focus on the mean time paths of aggregate variables such as GDP is not justified. It, in turn, means that such sophisticated microeconomic analyses as infinite horizon stochastic dynamic programming which are common in macroeconomic models, and are expected to give us the exact mean time paths of aggregate variables, have, in fact, little value. Those analyses provide us with no foundations for macroeconomic analyses because time paths of macro variables are sample dependent in any way.

Although we have discussed only the two-parameter Poisson-Dirichlet distribution, non-self averaging macroeconomic phenomena are observed in several other contexts, such as interacting two blocks of economies, or urn models used to model growth phenomena studied by Puyhaubert (2003), and Janson (2005).

References

[1] Aghion, P., and P. Howitt, (1992),"A model of growth through creative destruction" *Econometrica*, **60**, 323–351.

One- and Two-Parameter Poisson-Dirichelt Models", forthcoming *J. Econ.Dyn. Control*.

[2] ——, (2003), "Models with Random Exchangeable Structures and Coexistence of Several Types of Agents in the Long-Run: New Implementations of Schumpeter's Dynamics" Tech. Report, Research Initiative and Development, Chuo Univ. Tokyo, Dec. 2003.

[3] ——, (2002), *Modeling Aggregate Behavior and Fluctuations in Economics: Stochastic Views of Interacting Agents* , Cambridge Univ. Press, New York.

[4] ——-, (2000a), "Open models of share markets with two dominant types of participants, ", *J.Econ. Behav. Org.* **49** 199-216.

[5] —-, (2000b), "Cluster size distributions of economic agents of many types in a market", *J. Math Anal. Appl*, **249**,32-52.

[6] —— (1996), *New Approaches to Macroeconomic Modeling: Evolutionary Stochastic Dynamics, Multiple Equilibria, and Externalities as Field Effects* , Cambridge Univ. Press, New York.

[7] —–, and H. Yoshikawa (2007), *Reconstructing Macroeconomics: A Perspective from Statistical Physics and Combinatorial Stochastic Processes* , Cambridge University Press, New York.

[8] Bingham, N.H., C. M. Goldie, and J. L. Teugels (1987), *Regular variation*, Cambridge Univ. Press, Cambridge, UK.

[9] Blumenfeld, R., and B. B. Mandelbrot, (1997), "Lévy dusts, Mittag-Leffler statistics, mass fractal lacunarity, and perceived dimension," *Phy. Rev. E* **56** 112-118

[10] Charalambides, A. C., *Enumerative Combinatorics* , Chapman Hall/CRC, 2002, london

[11] Carlton, M. A. (1999) *Applications of the Two-Parameter Poisson-Dirichlet Distribution* Ph.D. thesis, Dept. Math. Univ. California, Los Angeles

[12] Devroye, L., (1993), "A triptych of discrete distributions related to the stable law," *Statist. Probab. Lett* **18**, 349–351

[13] Erdélyi, A. (ed.) (1955) *Higher Transcendental Functions, vol 3.*, McGraw-Hill, New York.

[14] Ewens, W. J. (1972), "The sampling theory of selectively neutral alleles," *Theor. Pop. Biol.*, **3**, 87-112.

[15] Feng, S., and F.M. Hoppe, (1998), " Large deviation principles for some random combinatorial structures in population genetics and brownian motion," *Ann. Appl. Probab.* **8**, 975–994.

[16] Grossman, G.M., and E. Helpman, (1991), *Innovation and Growth in the Global Economy,* MIT Press, Cambridge, MA.

[17] Janson, S. (2006), "Limit Theorems for Triangular Urn Schemes," Working Paper.

[18] Karlin, S., and J. McGregor, (1967), "The number of mutant froms maintained in a population," Proc. 5th Berkeley Symp. on Math. Stat. and Probab. vol 4.

[19] Kingman, J.F.C. (1978), "The representtion of partition structure", *J. London Math. Soc.*,**18**, 374–380.

[20] ——, (1993), *Poisson Processes*, Clarendon Press, Oxford UK.

[21] Kydland, F. and E. Prescott, (1982), "Time to Build and Aggregate Flucutaions," *Econometrica,* November.

[22] Lucas, R., (1988), "On the Mechanics of Economic Development," *Journal of Monetary Economics,* July.

[23] Pitman, J., (2002), *Lecture Notes of the summer school on probability* , St. Flour, France, forthcoming from Springer-Verlag.

[24] Plosser, C., (1989), "Understanding Real Business Cycles," *Journal of Economic Perspectives,* Summer.

[25] Puyhaubert, V., (2003), " Analytic Urns of Triangular Form" in INRIA F.Chyzk (ed) Algorithm Seminar, 2002-2004, 2005, pp.61-64.

[26] Romer, P.M.,(1986), "Increasing Returns and Long Run Growth," *Journal of Political Economy.*

[27] Solow, R. M. (2000), *Growth Theory 2nd ed.*, Oxford University Press:New York.

[28] Yamato, H., and M. Sibuya, (2000), "Moments of some statistics of Pitman sampling formula", *Bull. Inform. Cybernetics*, **6**, 463-488.

[29] ——(2003) "Some topics on Pitman's probabilistic partition" *Statistical Math.***51**, 351–372. (in japanese)